JOIN-SEMIDISTRIBUTIVE LATTICES OF RELATIVELY CONVEX SETS

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Abstract. We give two sufficient conditions for the lattice $\mathrm{Co}(\mathbb{R}^n, X)$ of relatively convex sets of $\mathbb{R}^n$ to be join-semidistributive, where X is a finite union of segments. We also prove that every finite lower bounded lattice can be embedded into $\mathrm{Co}(\mathbb{R}^n, X)$, for a suitable finite subset X of $\mathbb{R}^n$.

1. Introduction

A lattice L is *join-semidistributive*, if

$$x \vee y = x \vee z \text{ implies that } x \vee y = x \vee (y \wedge z),$$

for all $x, y, z \in L$. Let $X \subseteq \mathbb{R}^n$, and let $\mathrm{Co}(\mathbb{R}^n, X)$ denote the lattice of convex subsets of $\mathbb{R}^n$ *relative to* X, that is,

$$\mathrm{Co}(\mathbb{R}^n, X) = \{\, Y \subseteq \mathbb{R}^n \mid Y = \mathrm{Co}(Y) \cap X \,\},$$

where $\mathrm{Co}(Y)$ denotes *the convex hull* of Y, for any $Y \subseteq \mathbb{R}^n$. For all $X \subseteq \mathbb{R}^n$, the closure operator $\phi\colon \mathcal{B}_{\mathbf{X}} \to \mathcal{B}_{\mathbf{X}}$, where $\phi(Y) = \mathrm{Co}(Y) \cap X$ for all $Y \subseteq \mathbb{R}^n$, satisfies the so-called *anti-exchange axiom* that makes lattices of relatively convex sets just another example of *a convex geometry* (see the extensive monograph [7], also [2]). It is well known (cf. [2]) that a finite convex geometry is join-semidistributive, whence the lattice $\mathrm{Co}(\mathbb{R}^n, X)$ is join-semidistributive, for any finite $X \subseteq \mathbb{R}^n$.

Problem 3 in [2] asks about a description of lattices embeddable into lattices of the form $\mathrm{Co}(\mathbb{R}^n, X)$ with finite X. Since any sublattice of a join-semidistributive lattice is join-semidistributive itself, all those lattices must also be join-semidistributive. Although the current paper does not provide a solution of the problem, it suggests some approaches to it. The main idea is to consider a more general setting for the problem dropping the requirement for X to be finite.

For a lattice L with the least element 0_L, let $\mathrm{At}(L)$ denote the set of *atoms* of L, that is, $\mathrm{At}(L) = \{\, x \in L \mid 0_L \prec x \,\}$. While finite convex geometries are always join-semidistributive, a convex geometry L satisfies a weaker property:

$$x \vee y = x \vee z \text{ implies that } x \vee y = x \vee (y \wedge z),$$

for all $x \in L$ and all $y, z \in \mathrm{At}(L)$. In other words, if $x \vee y = x \vee z$, for some $x \in L$ and $y, z \in \mathrm{At}(L)$ the either $y = z$ or $y, z \leq x$. How weak this property is can be seen from the following result established in [4]: *every finite lattice can be embedded into* $\mathrm{Co}(\mathbb{R}^n, X)$*, for some* $n \in \omega$ *and* $X \subseteq \mathbb{R}^n$. Thus we would like to generalize

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Problem 3 from [2], dropping the requirement for X to be finite but still assuming $\mathrm{Co}(\mathbb{R}^n, X)$ to be join-semidistributive:

Problem 1. Which finite lattices can be embedded into join-semidistributive lattices of the form $\mathrm{Co}(\mathbb{R}^n, X)$?

It turns out that sets X for which the corresponding lattice $\mathrm{Co}(\mathbb{R}^n, X)$ is join-semidistributive are quite specific. The third section of the paper is mostly devoted to the case when X is a finite union of segments, which seems to be a natural generalization of finiteness of X. We provide two sufficient conditions for X to ensure $\mathrm{Co}(\mathbb{R}^n, X)$ to be join-semidistributive.

The last section is devoted to an important proper subclass of the class of join-semidistributive lattices, the class of so-called *lower bounded lattices*. We prove that every finite lower bounded lattice embeds into a finite lower bounded lattice of the form $\mathrm{Co}(\mathbb{R}^n, X)$. Another proof of this result can be found also in [10].

Here we use an essentially geometric idea, first constructing an embedding of the lattice $\mathrm{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$ of meet-subsemilattices of the Boolean lattice $\mathcal{B}_{\mathbf{n+1}}$ into the lattice of bounded convex subsets of $\mathbb{R}^n$, and then finding a finite set X which provides an embedding into $\mathrm{Co}(\mathbb{R}^n, X)$. We hope that this construction might give some additional insight into the question whether every finite join-semidistributive lattice embeds into a finite lattice $\mathrm{Co}(\mathbb{R}^n, X)$.

2. Basic concepts

For any $a, b \in \mathbb{R}^n$, let (a, b) denote the open segment and let $[a, b]$ denote the closed segment whose end points are a and b, that is,

$$(a, b) = \{\, x \in \mathbb{R}^n \mid x = \lambda a + (1-\lambda)b \text{ for some } \lambda \in (0,1) \,\},$$
$$[a, b] = \{\, x \in \mathbb{R}^n \mid x = \lambda a + (1-\lambda)b \text{ for some } \lambda \in [0,1] \,\}.$$

It is straightforward to verify that for any $Y \subseteq \mathbb{R}^n$,

$$\mathrm{Co}(Y) = \bigcup_{i \in \omega} Y^{(i)},$$

where $Y^{(0)} = Y$ and $Y^{(i+1)} = \{\, [a, b] \mid a, b \in Y^{(i)} \,\}$, for all $i \in \omega$.

A convex subset $F \subseteq P$ of a convex potytope P is *a face of* P, if $(a, b) \cap F \neq \varnothing$ implies $[a, b] \subseteq F$, for all $a, b \in P$. An element x of a convex set $X \subseteq \mathbb{R}^n$ is *an extreme point* of X if $x \notin \mathrm{Co}(X \setminus \{\, x \,\})$. Let $\mathrm{Ex}(X)$ denote the set of extreme points of X, for any $X \in \mathrm{Co}(\mathbb{R}^n)$.

For any $Y \subseteq \mathbb{R}^n$, we denote by $\overline{Y}$ the closure of Y and by $int_n(Y)$ the interior of Y in the Euclidean topology of $\mathbb{R}^n$.

Lemma 2.1. *Let $X \subseteq \mathbb{R}^n$ be a finite union of segments. Then $\mathrm{Co}(\overline{X}) = \overline{\mathrm{Co}(X)}$. In particular, if $x \in \mathrm{Ex}(\overline{\mathrm{Co}(X)})$ then x is an extreme point of a closure of a segment from X.*

Proof. The proof is straightforward. □

Lemma 2.2. *Let $P \subseteq \mathbb{R}^n$ be a convex polytope and let F be a face of P. Then $\mathrm{Co}(Y) \cap F = \mathrm{Co}(Y \cap F)$, for any $Y \subseteq P$.*

Proof. By induction on k, we prove that $Y^{(k)} \cap F \subseteq (Y \cap F)^{(k)}$, for all $k \in \omega$. For $k = 0$, the conclusion is obvious. Let $k > 0$ and let $x \in Y^{(k)} \cap F$. Then there

exist $a, b \in Y^{(k-1)}$ such that $x \in [a, b]$. If $x = a$ or $x = b$, then $x \in Y^{(k-1)} \cap F \subseteq (Y \cap F)^{(k-1)}$ by the induction hypothesis. Otherwise, $x \in (a, b) \cap F$, whence $a, b \in F$ since F is a face of P. Therefore, $a, b \in Y^{(k-1)} \cap F \subseteq (Y \cap F)^{(k-1)}$ by the induction hypothesis, whence $x \in (Y \cap F)^{(k)}$. □

For any $Y \subseteq \mathbb{R}^n$, let $\psi_Y \colon \operatorname{Co}(\mathbb{R}^n) \to \operatorname{Co}(\mathbb{R}^n, Y)$ be the map defined by $\psi_Y(X) = X \cap Y$, for any $X \in \operatorname{Co}(\mathbb{R}^n)$. Then ψ_Y preserves meets, for any $Y \subseteq \mathbb{R}^n$.

Lemma 2.3. *Let P be a convex polytope and let $X \subseteq P$. Then the map $\psi_F \colon \operatorname{Co}(\mathbb{R}^n, X) \to \operatorname{Co}(\mathbb{R}^n, X \cap F)$ defined by $\psi_F(Y) = Y \cap F$ is a surjective lattice homomorphism, for any face F of P.*

Proof. The surjectivity of ψ_F follows from the fact that if $A = \operatorname{Co}(A) \cap X \cap F$ then $A = \psi_F(\operatorname{Co}(A) \cap X)$. Let $A, B \in \operatorname{Co}(\mathbb{R}^n, X)$. Evidently, ψ_F preserves meets. Applying Lemma 2.2 we get

$$\begin{aligned}\psi_F(A \vee B) &= \operatorname{Co}(A \cup B) \cap X \cap F = \operatorname{Co}\big((A \cap F) \cup (B \cap F)\big) \cap X \\ &= \big(\operatorname{Co}(A \cap F) \cap X\big) \vee \big(\operatorname{Co}(B \cap F) \cap X\big) = \psi_F(A) \vee \psi_F(B),\end{aligned}$$

whence ψ_F preserves joins. □

3. Join-semidistributivity of $\operatorname{Co}(\mathbb{R}^n, X)$

If $X \subseteq \mathbb{R}^n$ is finite, then, as we mentioned above, the lattice $\operatorname{Co}(\mathbb{R}^n, X)$ is a finite convex geometry; in particular, it is join-semidistributive. However, we do not know how far this fact can be extended.

Problem 2. Describe sets $X \subseteq \mathbb{R}^n$ such that the lattice $\operatorname{Co}(\mathbb{R}^n, X)$ is join-semidistributive.

To remind that not every X suits, we recall an example given in [4].

Example 3.1. Let X contain the (2-dimensional) interior of some triangle TML. Pick any point K inside that interior. Then the interior of each triangle TMK, TLK, and MLK belongs to $\operatorname{Co}(\mathbb{R}^n, X)$, and they form a modular sublattice isomorphic to M_3. In particular, $\operatorname{Co}(\mathbb{R}^n, X)$ is not join-semidistributive.

A subset X of $\mathbb{R}^n$ is *sparse*, if $int_2(X \cap H) = \varnothing$, for any 2-dimensional affine subspace H of $\mathbb{R}^n$. From Example 3.1, it follows that every set X satisfying the requirement of Problem 2 has to be sparse.

Observe that if X is a line in $\mathbb{R}^n$ then $\operatorname{Co}(\mathbb{R}^n, X)$ is isomorphic to $\operatorname{Co}(\mathbb{R})$, the lattice of order convex subsets of $\mathbb{R}$, and the latter is join-semidistributive (see Theorem 14 in [5]).

Another extreme case is when X is the boundary of a ball; in this case, the lattice $\operatorname{Co}(\mathbb{R}^n, X)$ is Boolean (cf. an example of section 9 in [4]); in particular, it is distributive. This gives two natural examples of sparse sets which qualify for Problem 2. Unfortunately, being a sparse set is a necessary condition but not sufficient.

Example 3.2. Let X be the union of three lines A, B, and C which are on the same plane and have a common intersection. Then $A \vee B = A \vee C = X$ but $A \vee (B \cap C) = A$ in $\operatorname{Co}(\mathbb{R}^n, X)$.

On the other hand, if we take *segments* instead of lines, then the corresponding lattice turns out to be join-semidistributive. Thus the following question is rather natural: *if X is a finite union of segments, is the lattice* $\mathrm{Co}(\mathbb{R}^n, X)$ *join-semidistributive?* Unfortunately, even this simplest generalization of finiteness of X does not ensure that $\mathrm{Co}(\mathbb{R}^n, X)$ is join-semidistributive, as the example below demonstrates.

Example 3.3. Let T be a triangle in $\mathbb{R}^2$ with the set of extreme points $\{ a, b, c \}$ and let $p, m \in int_2 T$, $p \neq m$. Without loss of generality, we may assume that p, m, and a are not collinear. We put $X = [b, c] \cup [p, a] \cup [m, a]$ and $A = [b, c]$, $B = (p, a)$, $C = (m, a)$. Then $A \vee B = A \vee C = X \setminus \{ a \} \neq A \vee (B \wedge C) = A$ in $\mathrm{Co}(\mathbb{R}^2, X)$. Thus this lattice is not join-semidistributive.

We note that the failure of join-semidistributivity in the example above is due to the fact that closed segments $[p, a]$ and $[m, a]$ have a common point. Also, it is essential that (p, a) and (m, a) are subsetes of $int_2 T$. Were points p and m chosen, say, on faces $[a, b]$ and $[a, c]$ of the triangle T, respectively, the lattice $\mathrm{Co}(\mathbb{R}^n, X)$ would be join-semidistributive.

For the rest of this section, we assume X to be a finite union of segments. The following theorem provides two sufficient conditions for $\mathrm{Co}(\mathbb{R}^n, X)$ to be join-semidistributive. Each of them eliminates at least one condition that plays role in Example 3.3.

Theorem 3.4. *Let $n, k \in \omega$ and let $X = \bigcup \{ I_j \mid j < k \}$, where $I_j \subseteq \mathbb{R}^n$ is a segment, for all $j < k$. Consider the following two conditions:*

(i) *$\overline{I_s} \cap \overline{I_t} = \varnothing$, for all $s, t < k$, $s \neq t$;*
(ii) *there exists a convex polytope $P \subseteq \mathbb{R}^n$ such that for any $j < k$, I_j is a subset of a face of P.*

If X satisfies either (i) *or* (ii) *then the lattice* $\mathrm{Co}(\mathbb{R}^n, X)$ *is join-semidistributive.*

Proof. We agrue by induction on n. Let $n = 1$. For any $X \subseteq \mathbb{R}$, the lattice $\mathrm{Co}(\mathbb{R}, X)$ is the lattice of order-convex subsets of X endowed with the standard (linear) order, thus it is join-semidistributive (see [5, Theorem 14]).

Let $n > 1$. Suppose that X satisfies either (i) or (ii) and $A \vee B = A \vee C > A \vee (B \cap C)$, for some $A, B, C \in \mathrm{Co}(\mathbb{R}^n, X)$. Let $Y = \mathrm{Co}(A \vee (B \cap C))$. Then $B, C \nsubseteq Y$. We prove that there are a convex polytope Q and a face F of Q such that $B \cap F \nsubseteq Y$ and $Y \subseteq Q$.

Suppose first that X satisfies (i). By Lemma 2.1, we get

$$K = \overline{\mathrm{Co}(A \cup B)} = \mathrm{Co}(\overline{A \vee B}) = \mathrm{Co}(\overline{A \vee C}) = \overline{\mathrm{Co}(A \cup C)}.$$

If $K \nsubseteq \overline{Y}$, then there exists an extreme point $a \in \mathrm{Ex}(K)$ such that $a \notin \overline{Y}$. Since $\overline{A} \subseteq \overline{Y}$, by Lemma 2.1, $a \in \overline{B} \cap \overline{C}$ contradicting (i). Thus, $B \subseteq K \subseteq \overline{Y}$ but $B \nsubseteq Y$. Therefore, there exists a face F of $\overline{Y}$ such that $B \cap F \nsubseteq Y$. We take $Q = \overline{Y}$ in this case.

Suppose that X satisfies (ii). Since $B \nsubseteq Y$, there is a face F of P such that $B \cap F \nsubseteq Y$. We take $Q = P$ in this case.

By Lemma 2.3, the map $\psi_F \colon \mathrm{Co}(\mathbb{R}^n, X \cap Q) \to \mathrm{Co}(\mathbb{R}^n, X \cap Q \cap F)$ is a lattice homomorphism. Thus, $\psi_F(A) \vee \psi_F(B) = \psi_F(A) \vee \psi_F(C)$. Also, the lattice $\mathrm{Co}(\mathbb{R}^n, X \cap F)$ is isomorphic to the lattice $\mathrm{Co}(\mathbb{R}^m, X \cap F)$, where $m \in \omega$ is the dimension of an affine subspace of $\mathbb{R}^n$ containing F. Moreover, $X \cap F$ is a finite union of segments. By the induction hypothesis, the lattice $\mathrm{Co}(\mathbb{R}^m, X \cap F)$ is

join-semidistributive, whence

$$\begin{aligned} B \cap F = &\psi_F(B) \subseteq \psi_F(A \vee B) = \\ &\psi_F(A) \vee \big((\psi_F(B) \cap \psi_F(C))\big) = \\ &\psi_F\big(A \vee (B \cap C)\big) = \psi_F(Y) \subseteq Y, \end{aligned}$$

a contradiction. □

4. Lower bounded lattices as sublattices of finite $\mathrm{Co}(\mathbb{R}^n, X)$

In this section, we consider sublattices of lattices of the form $\mathrm{Co}(\mathbb{R}^n, X)$, where $X \subseteq \mathbb{R}^n$ is finite. As was observed in [2], we do not know yet any special type of finite convex geometries which admit any finite join-semidistributive lattice as a sublattice. We have a partial confirmation that lattices of the form $\mathrm{Co}(\mathbb{R}^n, X)$ could be such a "universal" class of convex geometries for the class of finite join-semidistributive lattices.

The main result of this section shows that, at least, this class is universal for the class of finite *lower bounded lattices* which is a proper subclass in the class of finite join-semidistributive lattices. We recall that a (finite) lattice is *lower bounded*, if it is an image of a finitely generated free lattice under *a lower bounded homomorphism*, that is, the preimage of every element under this homomorphism has a least element. We refer the reader to the comprehensive monograph on the topic [6]. There exist at least two other particular classes of finite convex geometries which admit every finite lower bounded lattice as a sublattice: suborder lattices of finite partial orders [9] and subsemilattice lattices of finite semilattices [1, 8].

Unlike these known examples, lattices of relatively convex subsets are *not* necessarily lower bounded. The simplest example is $\mathrm{Co}(\mathbb{R}, X)$, where X consists of four different points on the same line. The other common feature of many types of convex geometries is that they are biatomic. Due to [5], a lattice L with the least element 0_L is *biatomic* if for any $x \in \mathrm{At}(L)$ and any $y, z \in \mathrm{At}(L)$, the inequality $x \leq y \vee z$ implies that there are $y', z' \in \mathrm{At}(L)$ such that $y' \leq y$, $z' \leq z$, and $x \leq y' \vee z'$.

A result from [3] shows that *not* every finite join-semidistributive lattice embeds into a finite biatomic join-semidistributive lattice. The counter-example from [3] is the lattice $\mathrm{Co}(\mathbb{R}^2, X)$, where X is a 5-element set of points on a plane. In particular, this emphasizes that lattices of relatively convex subsets are essentially non-biatomic, thus might serve as a "universal" class of convex geometries for the class of finite join-semidistributive lattices.

Observe that an alternate approach which leads to the result that every finite lower bounded lattice is a sublattice of some $\mathrm{Co}(\mathbb{R}^n, X)$ with finite X is presented in [10]. The authors of [10] find an embedding of every finite lower bounded lattice into the lattice of convex polytopes of a finite-dimensional vector space, from where the result easily follows.

Proposition 4.1. *For every $n < \omega$, the lattice $\mathrm{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$ embeds into the lattice of bounded convex sets of $\mathbb{R}^n$.*

Proof. Let $S_{\mathbf{n+1}}$ denote a regular polytope in $\mathbb{R}^n$ with $n+1$ vertices. It is not that important to have a *regular* polytope, but it is easier to deal with because of the total symmetry of the argument. Thus, in $\mathbb{R}^2$ it is an equilateral triangle, in $\mathbb{R}^3$ it is a regular tetrahedron, etc.

Let $\mathrm{Ex}(S_{\mathbf{n+1}}) = \{\, p_i \mid i \leqslant n+1 \,\}$. We define the map $\psi\colon \mathcal{B}_{\mathbf{n+1}} \to \mathrm{Co}(\mathbb{R}^n)$ by the rule

$$\psi(t) = \begin{cases} \varnothing, \text{ if } t = \mathbf{n+1}, \\ \{\, p_i \,\}, \text{ if } \mathbf{n+1} \backslash t = \{\, i \,\}, \\ int_{|A|} \mathrm{Co}\big(\{\, p_i \mid i \in A = \mathbf{n+1} \backslash t \,\}\big), \text{ if } |t| < n. \end{cases} \tag{1}$$

Claim 1. *For any* $a, b \in \mathcal{B}_{\mathbf{n+1}}$, $\mathrm{Co}\big(\psi(a) \cup \psi(b)\big) = \psi(a) \cup \psi(b) \cup \psi(a \cap b)$.

Proof of Claim. Without loss of generality, we may assume that a and b are non-comparable. By induction on i, we prove that $\big(\psi(a) \cup \psi(b)\big)^{(i)} \subseteq \psi(a) \cup \psi(b) \cup \psi(a \cap b)$, for all $i \in \omega$. For $i = 0$, the conclusion is obvious. Suppose that $i < \omega$ and that $z \in \big(\psi(a) \cup \psi(b)\big)^{(i+1)} \backslash \big(\psi(a) \cup \psi(b)\big)^{(i)}$. Then there are $\lambda \in (0,1)$, $x, y \in \big(\psi(a) \cup \psi(b)\big)^{(i)}$ such that $z = \lambda x + (1-\lambda) y$. By the induction hypothesis, $x, y \in \psi(a) \cup \psi(b) \cup \psi(a \cap b)$. We consider several cases:

Case 1. $x, y \in \psi(a)$ or $x, y \in \psi(b)$. In this case, $z \in \psi(a) \cup \psi(b)$ since both $\psi(a)$ and $\psi(b)$ are convex.

Case 2. $x \in \psi(a)$ and $y \in \psi(b)$. In this case, there are $\lambda_k \in (0,1)$, $k \in \mathbf{n+1} \backslash a$, and $\mu_l \in (0,1)$, $l \in \mathbf{n+1} \backslash b$, such that

$$\begin{gathered} \sum \{\, \lambda_k \mid k \in \mathbf{n+1} \backslash a \,\} = \sum \{\, \mu_l \mid l \in \mathbf{n+1} \backslash b \,\} = 1 \text{ and} \\ x = \sum \{\, \lambda_k p_k \mid k \in \mathbf{n+1} \backslash a \,\}, \qquad y = \sum \{\, \mu_l p_l \mid l \in \mathbf{n+1} \backslash b \,\}. \end{gathered}$$

Then

$$z = \sum \{\, \lambda \lambda_k p_k \mid k \in \mathbf{n+1} \backslash a \,\} + \sum \{\, (1-\lambda) \mu_l p_l \mid l \in \mathbf{n+1} \backslash b \,\}.$$

Moreover, $\lambda \lambda_k$, $(1-\lambda)\mu_l \in (0,1)$, for all $k \in \mathbf{n+1} \backslash a$ and all $l \in \mathbf{n+1} \backslash b$, and

$$\sum \{\, \lambda \lambda_k \mid k \in \mathbf{n+1} \backslash a \,\} + \sum \{\, (1-\lambda) \mu_l \mid l \in \mathbf{n+1} \backslash b \,\} = \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.$$

Thus, $z \in \psi(a \cap b)$.

Case 3. $x \in \psi(a)$, $y \in \psi(a \cap b)$. In this case, there are $\lambda_k \in (0,1)$, $k \in \mathbf{n+1} \backslash a$, and $\mu_l \in (0,1)$, $l \in \mathbf{n+1} \backslash (a \cap b)$, such that

$$\begin{gathered} \sum \{\, \lambda_k \mid k \in \mathbf{n+1} \backslash a \,\} = \sum \{\, \mu_l \mid l \in \mathbf{n+1} \backslash (a \cap b) \,\} = 1 \text{ and} \\ x = \sum \{\, \lambda_k p_k \mid k \in \mathbf{n+1} \backslash a \,\}, \qquad y = \sum \{\, \mu_l p_l \mid l \in \mathbf{n+1} \backslash (a \cap b) \,\}. \end{gathered}$$

Then

$$z = \sum \{\, \big(\lambda \lambda_k + (1-\lambda) \mu_k\big) p_k \mid k \in \mathbf{n+1} \backslash a \,\} + \sum \{\, (1-\lambda) \mu_l p_l \mid l \in a \backslash b \,\}.$$

Again, all the coefficients are from $(0,1)$, and

$$\begin{aligned} &\sum \{\, \lambda \lambda_k + (1-\lambda) \mu_k \mid k \in \mathbf{n+1} \backslash a \,\} + \sum \{\, (1-\lambda) \mu_l \mid l \in a \backslash b \,\} = \\ &= \lambda \sum \{\, \lambda_k \mid k \in \mathbf{n+1} \backslash a \,\} + (1-\lambda) \sum \{\, \mu_l \mid l \in \mathbf{n+1} \backslash (a \cap b) \,\} = \\ &= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1. \end{aligned}$$

Thus, $z \in \psi(a \cap b)$. Therefore, we have proved that $\mathrm{Co}\big(\psi(a) \cup \psi(b)\big) \subseteq \psi(a) \cup \psi(b) \cup \psi(a \cap b)$.

We prove the inverse inclusion. It suffices to show that $\psi(a\cap b) \subseteq \operatorname{Co}\bigl(\psi(a)\cup\psi(b)\bigr)$. Let $z \in \psi(a \cap b)$. There are $\lambda_k \in (0,1)$, $k \in \mathbf{n+1}\backslash(a \cap b)$ such that $\sum\{\, \lambda_k \mid k \in \mathbf{n+1}\backslash(a \cap b)\,\} = 1$ and

$$z = \sum\{\, \lambda_k p_k \mid k \in \mathbf{n+1}\backslash(a \cap b)\,\}.$$

We put

$$\begin{aligned}
\lambda &= \Bigl(\sum\{\, \lambda_k \mid k \in b\backslash a\,\} + \frac{1}{2}\sum\{\, \lambda_k \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\}\Bigr)^{-1};\\
x &= \sum\{\, \frac{\lambda_k}{\lambda} p_k \mid k \in b\backslash a\,\} + \sum\{\, \frac{\lambda_k}{2\lambda} p_k \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\};\\
y &= \sum\{\, \frac{\lambda_k}{1-\lambda} p_k \mid k \in a\backslash b\,\} + \sum\{\, \frac{\lambda_k}{2(1-\lambda)} p_k \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\}.
\end{aligned}$$

We get

$$\begin{aligned}
&\sum\{\, \frac{\lambda_k}{\lambda} \mid k \in b\backslash a\,\} + \sum\{\, \frac{\lambda_k}{2\lambda} \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\} =\\
&=\frac{1}{\lambda}\Bigl(\sum\{\, \lambda_k \mid k \in b\backslash a\,\} + \frac{1}{2}\sum\{\, \lambda_k \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\}\Bigr) =\\
&=\frac{1}{\lambda}\cdot\lambda = 1;\\
&\sum\{\, \frac{\lambda_k}{1-\lambda} \mid k \in a\backslash b\,\} + \sum\{\, \frac{\lambda_k}{2(1-\lambda)} \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\} =\\
&=\frac{1}{1-\lambda}\Bigl(\sum\{\, \lambda_k \mid k \in a\backslash b\,\} + \frac{1}{2}\sum\{\, \lambda_k \mid k \in \mathbf{n+1}\backslash(a \cup b)\,\}\Bigr) =\\
&=\frac{1}{1-\lambda}\cdot(1-\lambda) = 1.
\end{aligned}$$

Thus, $x \in \psi(a)$ and $y \in \psi(b)$. Moreover, $z = \lambda x + (1-\lambda)y$, whence $z \in \operatorname{Co}\bigl(\psi(a) \cup \psi(b)\bigr)$. $\square$ Claim 1.

For any $S \in \operatorname{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$, we put

$$\varphi(S) = \bigcup\{\, \psi(t) \mid t \in S\,\}. \tag{2}$$

According to Claim 1, $\varphi(S) \in \operatorname{Co}(\mathbb{R}^n)$, for any $S \in \operatorname{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$. We verify that φ is a lattice homomorphism from $\operatorname{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$ to $\operatorname{Co}(\mathbb{R}^n)$. It is straighforward that φ is one-to-one. Moreover, φ preserves meets.

Let $S_0, S_1 \in \operatorname{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$ and let $S = S_1 \vee S_2$. If $t \in S\backslash(S_0 \cup S_1)$, then $t = t_0 \cap t_1$, for some $t_i \in S_i$, $i < 2$. Hence, by Claim 1, $\psi(t) \subseteq \operatorname{Co}\bigl(\psi(t_0) \cup \psi(t_1)\bigr) \subseteq \varphi(S_0) \vee \varphi(S_1)$. Thus $\varphi(S_0 \vee S_1) \subseteq \varphi(S_0) \vee \varphi(S_1)$, whence φ preserves joins. $\square$

For any $k < \omega$, for any $\lambda \geqslant 0$ small enough, and for any convex polytope $P \subseteq \mathbb{R}^k$, let P^λ denote the (nonempty) convex polytope which is a subset of P, whose faces are parallel to the corresponding faces of P, and $\rho(P^\lambda, P) = \lambda$, where $\rho(A,B)$ denotes the distance between A and B defined by the standard Euclidean metric ρ. For any $x \in \operatorname{Ex} P$, let x^λ denote the corresponding extreme point of P^λ.

We fix $n \in \omega$ and consider the polytope $S_{\mathbf{n+1}}$ defined in the proof of Proposition 4.1. Let $\lambda > 0$ be small enough.

If $A \subseteq \mathbf{n+1}$ and $|A| = k+1$, for some $k < \omega$, then S_A denotes the regular polytope in $\mathbb{R}^k$ with the set of extreme points $\operatorname{Ex} S_A = \{\, p_i \mid i \in A \,\}$. For any $B \subseteq A$, we put

$$H_B = \{ \sum_{i \in B} \lambda_i p_i^{\lambda} \mid \lambda_i \in \mathbb{R} \text{ for all } i \in B \}.$$

For any different $i, j \in A$, let $p(i, A, j)$ be a unique point from the intersection $[p_i, p_j] \cap H_{A \setminus \{ j \}}$. We put

$$T(A, \lambda, j) = \operatorname{Co}\bigl(\{\, p_i, p(i, A, j) \mid i \in A, i \neq j \,\}\bigr).$$

For any $j \in A$, the convex polytope $T(A, \lambda, j)$ has two parallel faces: one is the face $S_{A \setminus \{ j \}}$ of the polytope S_A, the other is the face $S'_{A \setminus \{ j \}} = \operatorname{Co}\bigl(\{\, p(i, A, j) \mid i \in A, i \neq j \,\}\bigr)$.

Lemma 4.2. *For any $j \in A$, $T(A, \lambda, j) \cap S_A^{\lambda} \subseteq S'_{A \setminus \{ j \}}$.*

Proof. The proof is straightforward. □

We also put $U(A, \lambda, i) = \operatorname{Co}\bigl(\{\, p_i \,\} \cup \{\, p(i, A, j) \mid j \in A, j \neq i \,\}\bigr)$.

Lemma 4.3. *For any $i \in A$, $U(A, \lambda, i) \subseteq \bigcap \{\, T(A, \lambda, j) \mid j \in A, j \neq i \,\}$.*

Proof. For any $j \in A$, $j \neq i$, the polytope $T(A, \lambda, j)$ contains the point p_i and the point $p(i, A, j)$. Moreover, it contains the whole face $S_{A \setminus \{ j \}}$ whence all the points $p(i, A, k)$, $k \neq i, j$. Therefore, $U(A, \lambda, i) \subseteq T(A, \lambda, j)$, for all $j \in A$, $j \neq i$. □

Lemma 4.4. *For any $i, j \in A$ such that $i \neq j$, $U(A, \lambda, i) \cap S'_{A \setminus \{ j \}} = \{\, p(i, A, j) \,\}$.*

Proof. $p(i, A, j) \in U(A, \lambda, i) \cap S'_{A \setminus \{ j \}}$ by the definition of $U(A, \lambda, i)$ and $S'_{A \setminus \{ j \}}$. To prove the reverse inclusion, we suppose that $z \in U(A, \lambda, i) \cap S'_{A \setminus \{ j \}}$. Then there are $\mu_j \in [0, 1]$, $j \in A$, such that $\sum \{\, \mu_j \mid j \in A \,\} = 1$ and $z = \mu_i p_i + \sum \{\, \mu_j p(i, A, j) \mid j \in A, j \neq i \,\}$. Since $S'_{A \setminus \{ j \}}$ is a face and $p_i \notin S'_{A \setminus \{ j \}}$, we have $\mu_i = 0$ and

$$\{\, p(i, A, j) \mid j \in A, j \neq i, \mu_j \neq 0 \,\} \subseteq S'_{A \setminus \{ j \}}.$$

Obviously, $p(i, A, k) \notin S'_{A \setminus \{ j \}}$, for all $k \neq i, j$. Thus, $\mu_k = 0$, for all $k \neq i, j$, whence $\mu_j = 1$ and $z = p(i, A, j)$. □

Lemma 4.5. *If $q_i \in U(A, \lambda, i) \setminus \{\, p(i, A, j) \mid j \in A, j \neq i \,\}$, for all $i \in A$, then $S_A^{\lambda} \subseteq int_{|A|} \operatorname{Co}\bigl(\{\, q_i \mid i \in A \,\}\bigr)$.*

Proof. For any $i \in A$, we put $B_i = \operatorname{Co}\bigl(\{\, q_j \mid j \in A, j \neq i \,\}\bigr)$. Then $B_i \subseteq T(A, \lambda, i)$, for all $i \in A$, by Lemma 4.4. Moreover, if $B_i \cap S'_{A \setminus \{ i \}} \neq \varnothing$, then there extsts $j \in A \setminus \{ i \}$ such that $q_j \in S'_{A \setminus \{ i \}} \cap U(A, \lambda, j)$ since $S'_{A \setminus \{ i \}}$ is a face of $T(A, \lambda, i)$. By Lemma 4.4, this implies that $q_j = p(j, A, i)$, a contradiction with the choice of q_j. Therefore, $B_i \subseteq T(A, \lambda, i) \setminus S'_{A \setminus \{ i \}}$.

By Lemma 4.2, we get $S_A^{\lambda} \cap B_i = \varnothing$, for all $i \in A$. Thus, for any $i \in A$, S_A^{λ} is a subset of the open half-space X_i defined by the hyperplane which contains B_i. Hence, $S_A^{\lambda} \subseteq \bigcap \{\, X_i \mid i \in A \,\} = int_{|A|} \operatorname{Co}\bigl(\{\, q_i \mid i \in A \,\}\bigr)$. □

Lemma 4.6. *There is $\varepsilon(\lambda) > 0$ such that $S_A^{\lambda} \subseteq int_{|A|} \operatorname{Co}\bigl(S^{\varepsilon}_{A \setminus \{ i \}} \cup S^{\varepsilon}_{A \setminus \{ j \}}\bigr)$, for any $\varepsilon \in (0, \varepsilon(\lambda)]$ and any $i, j \in A$, $i \neq j$.*

Proof. We pick $\varepsilon(\lambda) > 0$ with respect to the property that the extreme point $p_k^{\varepsilon(\lambda)}$ of the polytope $S_{A\setminus\{i\}}^{\varepsilon(\lambda)}$ (of the polytope $S_{A\setminus\{j\}}^{\varepsilon(\lambda)}$, respectively) belongs to $U(A,\lambda,k)$, for all $k \in A\setminus\{i\}$ (for all $k \in A\setminus\{j\}$, respectively). The desired conclusion follows then from Lemma 4.5. $\square$

We construct the finite set X which provides an embedding of the lattice $\mathrm{Sub}_\wedge\mathcal{B}_{\mathbf{n+1}}$ into the lattice $\mathrm{Co}(\mathbb{R}^n, X)$. Let v be the center of $S_{\mathbf{n+1}}$. Let $\lambda_0 > 0$ be small enough. Suppose that $k < n-1$ and we have already found $\lambda_0,\dots, \lambda_k > 0$ such that $\lambda_j \in (0, \varepsilon(\lambda_{j-1})]$, for all $0 < j \leqslant k$. By Lemma 4.6, there exists $\lambda_{k+1} \in (0, \varepsilon(\lambda_k)]$ such that, for any $A \subseteq \mathbf{n+1}$ with $|A| = n+1-k > 2$ and any $i, j \in A$, $i \neq j$, we have $S_A^{\lambda_k} \subseteq int_{|A|}\mathrm{Co}\big(S_{A\setminus\{i\}}^{\lambda_{k+1}} \cup S_{A\setminus\{j\}}^{\lambda_{k+1}}\big)$. We put $\lambda_n = 0$. For any nonempty $A \subseteq \mathbf{n+1}$ and any $i \in A$, we also put

$$P_A = S_A^{\lambda_k}, \qquad U(A,i) = U(A,\lambda_k,i), \qquad p(i,A) = p_i^{\lambda_k}$$

where $k < n+1$ is such that $|A| + k = n+1$.

Lemma 4.7. *For any $A \subseteq B \subseteq \mathbf{n+1}$ and any $i \in A$, we have $U(A,i) \subseteq U(B,i)$.*

Proof. We argue by induction on $|B\setminus A|$. If $|B\setminus A| = 0$ then $U(B,i) = U(A,i)$, and we are done. Let $j \in B\setminus A$. By the induction hypothesis, $U(A,i) \subseteq U(B\setminus\{j\}, i)$. All the extreme points of the polytope $U(B\setminus\{j\}, i)$ are in the interior of the face of $U(B,i)$ which is the convex hull of the set $\{p_i\} \cup \{p(i,B,k) \mid k \in B, k \neq i, j\}$. Therefore, $U(B\setminus\{j\}, i) \subseteq U(B,i)$. $\square$

We define the desired set X by

$$X = \{v\} \cup \bigcup \{\mathrm{Ex}\, P_A \mid A \subset \mathbf{n+1}\}.$$

First we notice the important property of the lattice $\mathrm{Co}(\mathbb{R}^n, X)$.

We remind that the *join dependency relation* D is defined for join irreducible elements a, b of a lattice L, $a\,D\,b$, if $a \neq b$, and there is a $p \in L$ with $a \leq b \vee p$ and $a \not\leq c \vee p$ for $c < p$. A D-sequence is a finite sequence $a_0, \dots, a_{n-1}$ ($n \geq 2$) of join irreducible elements of L such that $a_i\,D\,a_{i+1}$ for all $i < n$, where the subscripts are computed modulo n. It is well-known that a finite lattice L is lower bounded iff it contains no D-cycles (see, for example, Corollary 2.39 in [6]).

Lemma 4.8. *The finite lattice $\mathrm{Co}(\mathbb{R}^n, X)$ is lower bounded.*

Proof. If $a, b \in X\setminus\{v\}$, then there are $A, B \subseteq \mathbf{n+1}$ such that $a \in \mathrm{Ex}\, P_A$ and $b \in \mathrm{Ex}\, P_B$. In this case, $\{a\}\,D\,\{b\}$ implies that $|B| < |A|$. Moreover, $\{v\}\,D\,\{a\}$, for any $a \in X\setminus\{v\}$, and $\{a\}D\{v\}$ holds for no $a \in X$. Thus, the lattice $\mathrm{Co}(\mathbb{R}^n, X)$ does not contain a D-cycle whence it is lower bounded. $\square$

Secondly, we observe that the composition of ψ_X defined in section 2, and φ given by (2) is a a desired mapping of lattices.

Proposition 4.9. *The map $\psi_X\varphi\colon \mathrm{Sub}_\wedge\mathcal{B}_{\mathbf{n+1}} \to \mathrm{Co}(\mathbb{R}^n, X)$ is a lattice embedding.*

Proof. Since both ψ_X and φ preserve meets, the composition $\psi_X\varphi$ also does.

If $A \in B_0\setminus B_1$, for some $B_0, B_1 \in \mathrm{Sub}_\wedge\mathcal{B}_{\mathbf{n+1}}$, then $x \in \psi_X\varphi(B_0)\setminus\psi_X\varphi(B_1)$, where $x \in \mathrm{Ex}\, P_{\mathbf{n+1}\setminus A}$ in the case $A \subset \mathbf{n+1}$ and $x = v$ in the case $A = \mathbf{n+1}$. Therefore, the map $\psi_X\varphi$ is one-to-one.

To prove that $\psi_X\varphi$ preserves joins, it suffices to show that, for any noncomparable sets $A_0, A_1 \subseteq \mathbf{n+1}$,

$$\psi(A_0 \cap A_1) \cap X \subseteq \mathrm{Co}\big(\psi(A_0) \cup \psi(A_1)\big) \cap X,$$

where ψ is the map defined by (1). By the definition, we have

$$\psi(A_0 \cap A_1) \cap X = \mathrm{Ex}\, P_{A_0 \cup A_1} = \{\, p(i, A_0 \cup A_1) \mid i \in A_0 \cup A_1 \,\},$$

when $A_0 \cup A_1 \subset \mathbf{n+1}$, and

$$\psi(A_0 \cap A_1) \cap X = \{\, v \,\},$$

when $A_0 \cup A_1 = \mathbf{n+1}$. By Lemma 4.7, for any $j_i \in A_i$, $i < 2$, we have $p(j_i, A_i) \in U(A_i \cup \{\, j_{1-i} \,\}, j_i) \subseteq U(A_0 \cup A_1, j_i)$. Thus, by Lemma 4.5, we get

$$\begin{aligned}\psi(A_0 \cap A_1) \cap X &\subseteq \mathrm{Co}\big(\{\, p(i, A_0) \mid i \in A_0 \,\} \cup \{\, p(i, A_1) \mid i \in A_1 \,\}\big) \cap X \\ &= \mathrm{Co}\big(\psi(A_0) \cup \psi(A_1)\big) \cap X.\end{aligned}$$

Moreover, for any $A_0, A_1 \subseteq \mathbf{n+1}$ such that $A_0 \cup A_1 = \mathbf{n+1}$, we have that $v \in \mathrm{Co}\big(\psi(A_0) \cup \psi(A_1)\big)$. The proof of the lemma is complete. $\square$

Now we state the main result of this section.

Theorem 4.10. *For any finite lower bounded lattice L, there is $n \in \omega$ and a finite set $X \subseteq \mathbb{R}^n$ such that the lattice $\mathrm{Co}(\mathbb{R}^n, X)$ is lower bounded and L embeds into both $\mathrm{Co}(\mathbb{R}^n)$ and $\mathrm{Co}(\mathbb{R}^n, X)$.*

Proof. According to [1, 8], for any finite lower bounded lattice L, there is $n \in \omega$ such that L is isomorphic to a sublattice of $\mathrm{Sub}_\wedge \mathcal{B}_{\mathbf{n+1}}$. The desired conclusion follows from Propositoins 4.1 and 4.9. $\square$

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